Asymptotic and Bootstrap Inference for the Generalized Gini Indices

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**Abstract**

The Gini index represents a special case of the generalized Gini indices, which permit to choose a level of inequality aversion and to stress the different proportions of the income distribution. In order to apply these indices to income sample data, it is necessary to use reliable inferential procedures. In fact, also if often in income studies we have large samples for which the precision of estimates is not of primary interest, it has been noticed that, however, the standard errors are very high. Strengthened by these reasons, in this paper inferential procedures for generalized Gini indices are studied, specifically for S- and E-Gini indices, defined by means of the asymptotic distribution of their estimators and by the bootstrap method. To do this, the level of coverage of confidence intervals of the indices has been validated using Monte Carlo simulations, assuming as a model for the size distribution of incomes the generalized beta of the second kind, which is very flexible, with the ability to take a wide variety of shapes depending on particular values of its parameters.

*Key words*: S-Gini; E-Gini; Asymptotic distribution; Confidence intervals; t-bootstrap; Generalized beta distribution of the second kind; Monte Carlo experiment.

**1 Motivation**

The Gini index represents a special case of the generalized Gini indices, which permit to choose a level of inequality aversion and to stress the different proportions of the income distribution. Donaldson and Weymark (1980, 1983), Yitzhaki (1983) and Chacvakarty (1988) proposed two families of generalized Gini indices, the S- and the E-Gini. A characterization of income distributions in terms of generalized Gini indices has been obtained by Keiber and Kotz (2002) and the main fields of application of the E-Gini index can be found in Schechtman and Yitzhaki (2003).

In order to apply these indices to income sample data, it is necessary to use reliable inferential procedures. In fact, also if often in income studies we have large samples for which the precision of estimates is not of primary interest, it has been noticed that, however, the standard errors are very high (Maasoumi, 1997). Consequently, in empirical studies it is necessary to report tests, standard errors and confidence intervals, in order to avoid misleading conclusions. In this sense, in literature two kinds of inferential procedures have been developed for inequality indices based on the asymptotic distribution of estimators and on bootstrap methods. Asymptotic inference is actually well known for
a wide variety of measures, in particular for the Gini index (see Barret and Pendakur, 1995; Zitikis, 2002). Nevertheless, some studies have shown that with sample sizes not very high, confidence intervals built with bootstrap procedures have a precision in the level of coverage more accurate than intervals based on the standard approximation to the normal (Mills and Zandvakili, 1997; Palmitesta, Provasi and Spera, 2000; Blewen, 2002).

Strengthened by these reasons, in this paper inferential procedures for generalized Gini indices are studied, specifically for S- and E-Gini indices, defined by means of the asymptotic distribution of their estimators and by the bootstrap method. To do this, the level of coverage of confidence intervals of the indices has been validated using Monte Carlo simulations, assuming as a model for the size distribution of incomes the generalized beta of the second kind (McDonald and Xu, 1995; see also Parker, 1999), which is very flexible, with the ability to take a wide variety of shapes depending on particular values of its parameters.

The paper is organized as follows. After introducing the generalized Gini indices for a discrete population in the next section, we present in the following two sections the asymptotic distribution of their estimators and in the fifth section the bootstrap confidence intervals. Then, in section 6 we show the results of the Monte Carlo experiment. Conclusion remarks are left to the last section.

2 The generalized Gini indices of inequality

The generalized Gini indices are defined on a set of \( n \) incomes ordered in a nondecreasing order \( x_{1:n} \leq x_{2:n} \leq \ldots \leq x_{n:n} \). More precisely, the (relative) S-Gini inequality index is given by

\[
I_{R,\nu} = 1 - \frac{1}{n\nu} \sum_{i=1}^{n} ((n+i-1)\nu - (n-i)\nu) \frac{x_{i:n}}{\mu},
\]

where \( \mu \) is the mean income and \( \nu \geq 1 \) is an aversion parameter to inequality (see Donaldson and Weymark, 1980; Yitzhaki, 1983). Varying the parameter \( \nu \), the (1) represents a family of S-Gini inequality indices, which are continuous, scale-free, S-concave, and invariant to arbitrary replications of the income vector (see Blackorby and Donaldson, 1978; Donaldson and Weymark, 1980).

The (relative) E-Gini index is given by

\[
I_{R,\delta} = 2 \left[ \frac{1}{n} \sum_{i=1}^{n} \left( \frac{i}{n} - \frac{1}{n\mu} \sum_{j=1}^{i} x_{j:n} \right)^{\delta} \right]^{1/\delta},
\]

where \( \delta \geq 1 \) is an aversion parameter to inequality (see Chakravarty, 1988). Varying the parameter \( \delta \), the (2) represents a family of E-Gini inequality indices with properties similar to those of the S-Gini family indices.

Note that the two sets of indices are identical if \( \nu = 2 \) in the (1) and \( \delta = 1 \) in the (2), because they can be seen as the classical Gini index (Xu, 2000):

\[
I_R = \frac{2}{n^2 \mu} \sum_{i=1}^{n} i x_{i:n} - \frac{n+1}{n}.
\]
When $\delta \neq 2$ and $\nu \neq 1$, the two sets handle data in a different way. In fact, the S-Gini index can be expressed also with the ordinates of the Lorenz curve:

$$I_{R,\nu} = 1 - \sum_{i=1}^{n} c_{in} L \left( \frac{i}{n} \right),$$  \hspace{1cm} (3)

where

$$L \left( \frac{i}{n} \right) = \frac{1}{n \mu} \sum_{j=1}^{i} x_{j:n}$$

is the ordinate of the Lorenz curve computed in the point $i/n$ and the $c_{in}$ are defined as $c_{in} = ((n-i+1)\nu - 2(n-i) + (n-i-1)\nu)/n^{\nu-1}$ for $i = 1, 2, \ldots, n-1$, and $c_{nn} = 1/n^{\nu-1}$. Therefore, the (3) shows that the S-Gini index is a linear combination of the ordinates of the Lorenz curve with weights linked to the ranks of individual incomes. Likewise, the E-Gini index can be expressed in its turn in terms of the ordinates of the Lorenz curve:

$$I_{R,\delta} = 2 \left[ \frac{1}{n} \sum_{i=1}^{n} \left( \frac{i}{n} - L \left( \frac{i}{n} \right) \right)^{\nu} \right]^{1/\nu}.$$  \hspace{1cm} (4)

Nevertheless, the (4) highlights that the E-Gini index is a nonlinear function of the ordinates of the Lorenz curve with weights determined by the corresponding weights of the cumulative incomes. In Figure 1 we show the trend of the two indices computed on the sample of individual incomes registered by the Bank of Italy in 2002 ($n = 8010$) with different values of the parameters $\nu$ and $\delta$. Note that the values assumed by the S-Gini index varying the aversion parameter have a growth rate higher than the E-Gini index values.

![Figure 1: Trend of the S- and E-Gini indices computed on individual incomes registered by the Bank of Italy in 2002.](image-url)
In the next two sections we present the estimators of the two indices when we assume that the income distribution of a population can be represented by a positive continuous random variable.

3 S-Gini estimator

Let the income of a population be distributed as a random variable $X \geq 0$ with probability density function (pdf) $f$ and cumulative distribution function (cdf) $F$. If we assume that the mean $\mu = E(X)$ of $X$ is finite and positive, the classical Gini index can be defined as

$$I_F = 1 - \frac{2}{\mu} \int_0^1 (1 - t)F^{-1}(t)dt,$$  \hspace{1cm} (5)

where

$$F^{-1}(t) = \inf \{u : F(u) \geq t\}, \quad 0 < t < 1,$$ \hspace{1cm} (6)

indicates the quantile function of $F$. The S-Gini index is an extension of the $I_F$ index and it is defined by the equation

$$I_{F,\nu} = 1 - \frac{\delta}{\mu} \int_0^1 (1 - t)^{\nu - 1}F^{-1}(t)dt,$$ \hspace{1cm} (7)

where $\nu \geq 1$ is an aversion parameter to inequality. It is immediate to note that $I_{F,\nu} = 0$ if $\nu = 1$ and that the (7) can be seen as the (5) if $\nu = 2$.

Now, suppose that the iid random variables $X_1, X_2, \ldots X_n$ compose a simple random sample $\chi$ of size $n$ drawn by $X$ and let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the corresponding order statistics. Moreover, let the $F_n$ be the empirical cdf based on $X_1, X_2, \ldots X_n$ and $F_n^{-1}$ be the corresponding quantile function defined by the (6) with $F_n$ instead of $F$. Then, if in the formula (7) $\mu$ is replaced by the sample mean $\bar{X}$ of $X_1, X_2, \ldots X_n$ and $F_n^{-1}$ by its empirical counterpart $F_n^{-1}$, we obtain the natural estimator of the S-Gini index

$$I_{n,\nu} = 1 - \frac{\nu}{\bar{X}} \int_0^1 (1 - t)^{\nu - 1}F_n^{-1}(t)dt,$$

which can be also expressed by

$$I_{n,\nu} = 1 - \frac{\nu}{\bar{X}} \sum_{i=1}^n \left[ \left( 1 - \frac{i - 1}{n} \right)^{\nu} - \left( 1 - \frac{i}{n} \right)^{\nu} \right] X_{i:n}.$$ \hspace{1cm} (8)

It is clear that this expression is equivalent to the (1).

3.1 Asymptotic distribution

The equation (8) can be also written as

$$I_{n,\nu} = 1 - \nu \tau_n,$$

where $\tau_n$ assumes the form

$$\tau_n = \frac{\sum_{i=1}^n c_{in} X_{i:n}}{\sum_{i=1}^n d_{in} X_{i:n}},$$

with $c_{in} = (1 - (i - 1)/n)^\nu - (1 - i/n)^\nu$ and $d_{ni} = 1/n$ for $i = 1, 2, \ldots, n$. This last expression highlights that the statistic $\tau_n$ is given by the ratio of two statistics of $L$ type,
LogLogistic ($a$)  
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Table 1: The values of $I_{F,\nu}$ and, in brackets, $\sigma_{F,\nu}$ in the case of the loglogistic distribution.

then from Sendler (1979) we deduce that $I_{n,\nu}$ is a consistent estimator of $I_{F,\nu}$ and that the asymptotic distribution of $\sqrt{n}(I_{n,\nu} - I_{F,\nu})$ is normal with zero mean and variance

$$
\sigma^2_{F,\nu} = \left(\frac{\nu}{\mu}\right)^2 \int_0^1 \int_0^1 (\min(s,t) - st)J(s)J(t)dF^{-1}(s)dF^{-1}(t),
$$

provided that $E(X^2) < \infty$, where $J(u) = (1-u)^{\nu-1} - (1 - I_{F,\nu})/\nu^1$.

Due to the complexity of the mathematical expression which defines $\sigma^2_{F,\nu}$, it is difficult to determine its dependence from the parameter $\nu > 1$ when $F$ is general. For this reason we restrict our analysis to the loglogistic distribution with cdf

$$
F(x) = 1 - \frac{1}{1 + x^a}, \quad x > 0,
$$

where $a > 0$ is a shape parameter. As we will see in the following, the loglogistic distribution is a special case of the generalized beta distribution of the second kind.

In Table 1 we write the values of $I_{F,\nu}$ and, in brackets, $\sigma_{F,\nu}$, considering only values of $a > 2$ to assure the existence of the second moment. From these values it is clear that the asymptotic variance depends both on the cdf and the parameter $\nu$.

### 3.2 Estimation of the asymptotic variance

The results of the previous section can be used to build an approximated confidence interval for the S-Gini index. Since the statistic

$$
T_{I_{n,\nu}} = \sqrt{n} \frac{I_{n,\nu} - I_{F,\nu}}{\sigma_{F,\nu}}
$$

is asymptotically distributed as a normal with zero mean and unit variance, an approximate $(1 - 2\alpha)$ equal-tail double-sided confidence interval for $I_{F,\nu}$ is given by

$$
I_{n,\nu} \pm z_{1-\alpha} \frac{\sigma_{F,\nu}}{\sqrt{n}}
$$

where $z_{1-\alpha} = \Phi^{-1}(1 - \alpha)$, being $\Phi$ the cdf of the standard normal distribution.

---

1 Zitikis and Gastwirth (2002) write the asymptotic variance $\sigma^2_{F,\nu}$ of $I_{n,\nu}$ in a different way but equivalent to the (9).
The interval (11) cannot be used in practice, because the variance in general depends on the unknown cdf $F$ and, consequently, it must be estimated. Nevertheless, a consistent estimate of $\sigma_{F,\nu}^2$ can be obtained substituting in the equation (9) $F$ with the empirical cdf $F_n$. Then, we have that

$$\sigma_{n,\nu}^2 = \frac{\nu}{X^2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left( \min\left(\frac{i}{n}, \frac{j}{n}\right) - \frac{i}{n} - \frac{j}{n} \right) \left( 1 - \frac{i}{n} \right) \left( 1 - \frac{j}{n} \right)$$

$$\cdot \left( 1 - \frac{i}{n} \right)^{\nu-1} - \frac{1 - I_{n,\nu}}{\nu} \right) (X_{i+1:n} - X_{i:n})(X_{j+1:n} - X_{j:n})$$

is a strongly consistent estimator of $\sigma_{F,\nu}^2$. Therefore, substituting in the (11) $\sigma_{F,\nu}$ with $\sigma_{n,\nu}$, the interval

$$I_{n,\nu} \pm z_{1-\alpha} \frac{\sigma_{F,\nu}}{\sqrt{n}}$$

represents for the S-Gini index a confidence interval of approximated level $(1 - 2\alpha)$.

### 4 E-Gini estimator

As previously, we assume that the income of a population is distributed as a random variable $X \geq 0$ with pdf $f$ and cdf $F$. If the mean $\mu = E(X)$ of $X$ is finite and positive, the Gini index can also be written in a way equivalent to the (5) as

$$I_F = 2 \int_0^1 (t - L_F(t)) dt,$$  

where $L_F(t)$ indicates the Lorenz function which can be defined as follows (Pietra, 1915; Gastwirth, 1971):

$$L_F(t) = \frac{1}{\mu} \int_0^t F^{-1}(s) ds, \quad 0 \leq t \leq 1.$$  

(14)

The E(xtended)-Gini index is an extension of the (13) and it is given by the equation

$$I_{F,\delta} = 2 \left( \int_0^1 (t - L_F(t))^{\delta} dt \right)^{1/\delta},$$  

(15)

where $\delta \geq 1$ is an aversion parameter to the inequality. It is immediate to observe that $I_{F,\delta}$ is equal to the (13) if $\delta = 1$.

Still, suppose that the iid random variables $X_1, X_2, \ldots, X_n$ form a simple random sample $\chi$ of size $n$ drawn by $X$ and let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the corresponding order statistics. Moreover, let $F_n$ be the empirical cdf based on $X_1, X_2, \ldots, X_n$ and $F_n^{-1}$ be the corresponding quantile function defined by the (6) with $F_n$ instead of $F$. Then, if in the formula (14) $\mu$ is substituted with the sample mean $\bar{X}$ of $X_1, X_2, \ldots, X_n$ and $F_n^{-1}$ is substituted with its empirical counterpart $F_n^{-1}$, we obtain the empirical Lorenz function

$$L_n(t) = \frac{1}{\bar{X}} \int_0^t F_n^{-1}(s) ds, \quad 0 \leq t \leq 1,$$

from which it follows that a natural estimator of the E-Gini index is

$$I_{n,\delta} = 2 \left( \sum_{i=1}^n \frac{i}{n} - L_n\left(\frac{i}{n}\right) \right)^{1/\delta}.$$  

(16)

It is evident that a natural estimator of the E-Gini index is equivalent to the (2).
Table 2: The values of $I_{F,\delta}$ and, in brackets, $\sigma_{F,\delta}$ in the case of loglogistic distribution.

4.1 Asymptotic distribution

Zitikis (2003) has shown that $I_{n,\delta}$ is a consistent estimator of $I_{F,\delta}$; moreover, if $E(X^2) < \infty$, the asymptotic distribution of $\sqrt{n}(I_{n,\delta} - I_{F,\delta})$ is normal with zero mean and variance

$$
\sigma^2_{F,\delta} = \frac{4^\delta}{I^{2(\delta-1)}_{F,\delta}} \mu^2 \Delta^2_{F,\delta}, \tag{17}
$$

where

$$
\Delta^2_{F,\delta} = \int_0^1 \int_0^1 (\min(u, v) - u v) \psi(u) \psi(v) dF^{-1}(u) dF^{-1}(v) 
- 2\mu \left( \int_0^1 L_F(t)(t - L_F(t))^{\delta-1} dt \right) \left( \int_0^1 (u - L_F(u)) \psi(u) dF^{-1}(u) \right) 
+ \sigma^2_X \left( \int_0^1 L_F(t)(t - L_F(t))^{\delta-1} dt \right)^2,
$$

with $\sigma^2_X$ indicating the variance of $X$ and $\psi(u) = \int_u^1 (t - L_F(t))^{\delta-1} dt$.

Due to the complexity of the mathematical expression defining $\sigma^2_{F,\delta}$, it is difficult to determine its dependence on the parameter $\delta \geq 1$ when $F$ is general. For this reason, as previously, we restrict our analysis to the loglogistic distribution with shape parameter $a > 0$. In Table 2 the values of $I_{F,\delta}$ are shown and, in brackets, $\sigma_{F,\delta}$, considering only values of $a > 2$ to assure the existence of the second moment.

Also for the E-Gini index, from the values in the table it is evident how the asymptotic variance depends both on the cdf and the parameter $\delta$.

4.2 Estimation of the asymptotic variance

The results of the preceding section can be used to build an approximated confidence interval for the E-Gini index. Since the statistic

$$
T_{I_{n,\delta}} = \sqrt{n} \frac{I_{n,\delta} - I_{F,\delta}}{\sigma_{F,\delta}}, \tag{18}
$$

...
is asymptotically distributed as a normal with zero mean and unit variance, an approximate \((1 - 2\alpha)\) equal-tail double-sided confidence interval for \(I_{F,\nu}\) is given by

\[
I_{n,\delta} \pm z_{1-\alpha} \frac{\sigma_{F,\delta}}{\sqrt{n}},
\]

where \(z_{1-\alpha} = \Phi^{-1}(1 - \alpha)\), being \(\Phi\) the cdf of the standard normal distribution.

The interval (19) cannot be used in practice, because the variance depends on the unknown cdf \(F\) and, consequently, it must be estimated. Nevertheless, a consistent estimate of \(\sigma^2_{F,\delta}\) can be obtained substituting in the equation (17) \(L_F\) with the empirical Lorenz curve \(L_n\). Then, from Zitikis (2003) we have that

\[
\sigma^2_{n,\delta} = \frac{4\delta}{I_{n,\delta}^2} \Delta^2_{n,\delta}
\]

is a strongly consistent estimator of \(\sigma^2_{F,\delta}\), where

\[
\Delta^2_{n,\delta} = \sum_{l=1}^{n-1} \sum_{m=1}^{n-1} \left( \min\left(\frac{l}{n}, \frac{m}{n}\right) - \frac{l}{n} \right) A_{n,\delta}(l)(X_{l+1:n} - X_{l:n})(X_{m+1:n} - X_{m:n})
\]

\[ - 2\bar{X} B_{n,\delta} \sum_{l=1}^{n-1} \left( \frac{l}{n} - L_n\left(\frac{l}{n}\right) \right) A_{n,\delta}(l)(X_{l+1:n} - X_{l:n}) + S^2 B^2_{n,\delta},
\]

with \(A_{n,\delta}(\cdot)\) given by

\[
A_{n,\delta}(i) = \frac{1}{n} \sum_{k=1}^{n-1} \left( \frac{k}{n} - L_n\left(\frac{k}{n}\right) \right)^{\delta-1},
\]

for any \(i = 1, 2, \ldots, n - 1\), and \(B_{n,\delta}\) given by

\[
B_{n,\delta} = \frac{1}{n} \sum_{k=1}^{n-1} L_n\left(\frac{k}{n}\right) \left( \frac{k}{n} - L_n\left(\frac{k}{n}\right) \right)^{\delta-1}.
\]

\(\bar{X}\) and \(S^2\) are, respectively, the sample mean and the sample variance of \(X_1, X_2, \ldots X_n\). Therefore, substituting in the (19) \(\sigma_{F,\delta}\) with \(\sigma_{n,\delta}\), the interval

\[
I_{n,\delta} \pm z_{1-\alpha} \frac{\sigma_{F,\delta}}{\sqrt{n}}
\]

represents for the E-Gini index a confidence interval of approximated level \((1 - 2\alpha)\).

5 Bootstrap confidence intervals

Several empirical studies on inequality indices belonging to the Gini family have shown that the level of approximation to the exact dimension \(\alpha\) of the confidence intervals based on the asymptotic distribution depends on the speed of convergence to the normal (see, for example, Giorgi and Pallini, 1990; Giorgi and Provasi, 1995). Consequently, in inferential applications with sample size not very large, the interval estimates can be not much accurate. However, several theoretical papers show that the order of accuracy can be increased by the construction of nonparametric bootstrap confidence intervals (for example, see Efron and Tibshirani, 1993; see also Horowitz, 2001, for econometric applications of bootstrap). Bootstrap procedures to build confidence intervals for inequality indices have

In nonparametric bootstrap, a bootstrap resample \( \chi^* = \{X_1^*, X_2^*, \ldots, X_n^*\} \) is a sample of size \( n \) drawn randomly, with replacement, from the sample \( \chi = \{X_1, X_2, \ldots, X_n\} \). The observations in the resample are iid \( X_i^* \sim F_n \), where \( F_n \) denote the empirical distribution function. Therefore, using the statistics \( I_{n,\nu}^* \) obtained in (8) through repeated samples as a function of the bootstrap resampling, we can build t-bootstrap confidence intervals for the S-Gini index based on the (10). In bootstrap the unknown distribution of \( T_{I_{n,\nu}} \) is approximated by the distribution of its bootstrap version. The bootstrap statistic is given by

\[
T_{I_{n,\nu}}^* = \sqrt{n} \frac{I_{n,\nu}^* - I_{n,\nu}}{\sigma_{n,\nu}^*},
\]

where \( \sigma_{n,\nu}^2 \) is the bootstrap version of \( \sigma_{n,\nu} \) given by the (12). Simulation-based bootstrap estimates of the quantile of \( T_{I_{n,\nu}}^* \) can be obtained from the empirical distribution of the \( B \) resampled statistics \( \{T_{I_{n,\nu}}^{*b} = \sqrt{n}(I_{n,\nu}^* - I_{n,\nu})/\sigma_{n,\nu}^*, b = 1, 2, \ldots, B\} \).

A \((1 - 2\alpha)\) equal-tail double-sided bootstrap-t confidence interval for the S-Gini index \( I_{F,\nu} \) is

\[
\left\{ I_{n,\nu} - t_{I_{n,\nu}}^{(1-\alpha)} \frac{\sigma_{n,\nu}}{\sqrt{n}}, I_{n,\nu} - t_{I_{n,\nu}}^{(\alpha)} \frac{\sigma_{n,\nu}}{\sqrt{n}} \right\},
\]

where the bootstrap quantile estimates of \( T_{I_{n,\nu}} \) are given by \( t_{I_{n,\nu}}^{(\alpha)} \) such that

\[
\sum_{i=1}^B H \left( T_{I_{n,\nu}}^{*b} \leq t_{I_{n,\nu}}^{(\alpha)} \right) / B = \alpha,
\]

with \( H \) that denotes the standard indicator function.

Likewise for the E-Gini index \( I_{F,\delta} \), from the (18) the bootstrap statistics is given by

\[
T_{I_{n,\delta}}^* = \sqrt{n} \frac{I_{n,\delta}^* - I_{n,\delta}}{\sigma_{n,\delta}^*},
\]

where \( \sigma_{n,\delta}^2 \) is the bootstrap version of \( \sigma_{n,\delta}^2 \) given by the (20). Then, a \((1 - 2\alpha)\) equal-tail double-sided bootstrap-t confidence interval for \( I_{F,\delta} \) is

\[
\left\{ I_{n,\delta} - t_{I_{n,\delta}}^{(1-\alpha)} \frac{\sigma_{n,\delta}}{\sqrt{n}}, I_{n,\delta} - t_{I_{n,\delta}}^{(\alpha)} \frac{\sigma_{n,\delta}}{\sqrt{n}} \right\},
\]

where the bootstrap quantile estimates of \( T_{I_{n,\delta}}^* \) are given by \( t_{I_{n,\delta}}^{(\alpha)} \) such that

\[
\sum_{i=1}^B H \left( T_{I_{n,\delta}}^{*b} \leq t_{I_{n,\delta}}^{(\alpha)} \right) / B = \alpha,
\]

with the meaning of symbols seen before.

The coverage probability of the two confidence intervals converges to \((1 - 2\alpha)\) as \( n \to \infty \), because the sampling statistics \( I_{n,\nu}, \sigma_{n,\nu} \) and \( I_{n,\delta}, \sigma_{n,\delta} \) strongly converge to \( I_{F,\nu}, \sigma_{F,\nu} \) and \( I_{F,\delta}, \sigma_{F,\delta} \), respectively. Hall (1988) showed that the bootstrap-t limits are second-order accurate. DiCiccio and Efron (1992) that they are also second-order correct.
6 Monte Carlo experiment

This section presents the results of a Monte Carlo experiment designed to measure the coverage performance in finite samples of the confidence intervals introduced in the last sections for the S- and E-Gini indices. To do this, we have assumed as a model for the size distribution of incomes the generalized beta of the second kind (GB2) (McDonald and Xu, 1995), which is very flexible, with the ability to take a wide variety of shapes depending on particular values of its parameters. The GB2 nests many distributions as special or limiting cases, including, among others, the lognormal, Weibull, gamma and loglogistic. The details of many of these relationships are summarized in McDonald (1984).

The pdf of the GB2 distribution with scale parameter \( b > 0 \) and shape parameters \((a,p,q) > 0\) can be written as

\[
f(x) = \frac{a}{b^p B(p,q)} x^{ap-1} \left(1 + \left(\frac{x}{b}\right)^a\right)^{-(p+q)} \quad \text{for } 0 < x < \infty, \tag{21}\]

and zero otherwise, where \( B(\cdot, \cdot) \) indicates the complete beta function. In Xu and Mac-Donald (1995) it is shown that the GB2 distribution can be expressed in terms of the beta distribution. Therefore, the cdf and the quantile distribution of \( X \) can be approximated by means of the several algorithms of numerical computation prepared for the incomplete beta distribution (cf. Balakrishnan, Johnson and Kotz, 1995)\(^2\).

The moments from the origin of order \( r, r = 1, 2, \ldots \), exist when \( aq > r \) and they are

\[
E(X^r) = \frac{b^r B(p + \frac{r}{2}, q - \frac{r}{2})}{B(p, q)}.
\]

Therefore, the inferential procedures relative to the two indices are applicable only when \( aq > 2 \). In the following we will refer to Monte Carlo drawn from a GB2 distribution with scale parameter equal to 1 and shape parameters equal to \( a = 2.5, p = 0.5 \) and \( q = 2 \), which closely mirrors the values reported in Table 1 presented in MacDonald and Xu (1995)\(^3\).

At the beginning we have done an analysis on the speed of convergence to the normal of the two sampling statistics given by (1) and (2) using the Kolmogorov distance. The results are shown in Tables 3 and 4, where we show the Kolmogorov distances of sampling indices from the distribution \( N(0, 1) \) obtained by \( N = 100, 000 \) independent samples with \( \nu = 1.5, 2.0, 2.5, 3.0, 4.0, 5.0 \) for the S-Gini index, \( \delta = 1, 2, 3, 5, 7, 10 \) for the E-Gini index and \( n = 25, 50, 75, 100, 150, 200, 300, 400, 500, 1000 \). To better interpret the results given in the two tables, we have displayed the trend of Kolmogorov distances varying the sample size in Figures 2 and 3. It is clear that the speed of convergence to the normal of the sampling distributions is influenced by the value of the parameters of inequality aversion, particularly for the S-Gini index.

Then we have considered the estimated coverage probability of intervals with nominal confidence intervals of 80%, 90%, 95% and 99% for samples of dimension \( n = 25, 50, 75, 100, 150 \) and 200 based on the asymptotic distribution and on the bootstrap-t. The results relative to the two indices obtained with \( N = 10,000 \) simulations for each scenario are given in Tables 5 and 6. The bootstrap intervals are built with \( B = 2000 \)

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\(^2\)The result of this application are obtained using a FORTRAN 90 code implemented on a 3200 MHz PC Intel with Windows XP. The random number generator routines are from the NAG Fortran library.

\(^3\)The parameters considered are the maximum likelihood estimates of the GB2 distribution based on 1985 nominal family income obtained from the Census Population Reports.
resamples (the various confidence interval types are computed from the same simulated data).

From these results we infer that the accuracy of the empirical level of confidence intervals based on the approximation to the normal distribution decreases as values of the parameters of inequality aversion decrease, in particular for the S-Gini index, because of the non uniformity of the speeds of convergence of sampling statistics; moreover, in general, the empirical level of the intervals based on the bootstrap-t is very close to the nominal level also with small samples. Therefore, we can conclude that it is preferable to use bootstrap inferential procedures for the application of generalized Gini indices to income data with small samples.

7 Conclusions

In this paper we have studied in finite samples the confidence intervals of the generalized Gini indices. Specifically the S- and E-Gini indices, by means of the asymptotic distribution of their natural estimators and the bootstrap method. The level of coverage of intervals has been validated using Monte Carlo simulations, assuming as a model for the size distribution of incomes the generalized beta of the second kind.

Our results show that the speed of convergence to the normal of sampling distributions is not uniform, because it depends on the value of the two indices. Therefore, with small sample sizes it could be better to use a bootstrap approach which guarantees a better level of approximation to the nominal confidence intervals. It’s in progress a paper on the empirically accelerated convergence to the normal of the sampling inequality indices as in Palmitesta (1996).
References


Table 3: Kolmogorov distance from the standard normal of the sampling distribution of the S-Gini index.

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Table 4: Kolmogorov distance from the standard normal of the sampling distribution of the E-Gini index.

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Figure 2: Trend of the Komogorov distance for the S-Gini index.

Figure 3: Trend of the Komogorov distance for the E-Gini index.
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<td>0.81</td>
<td>0.91</td>
<td>0.96</td>
<td>0.99</td>
</tr>
</tbody>
</table>

Table 5: Monte Carlo confidence levels for the S-Gini index.
| $\nu$ | $I_{F,\delta}$ | $n$ | \(0.80\) & \(0.90\) & \(0.95\) & \(0.99\) & \(0.80\) & \(0.90\) & \(0.95\) & \(0.99\) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.3981 | 25 | 0.72 & 0.83 & 0.89 & 0.95 & 0.79 & 0.90 & 0.95 & 0.99 |
| | | 50 | 0.76 & 0.86 & 0.91 & 0.97 & 0.79 & 0.90 & 0.95 & 0.99 |
| | | 75 | 0.76 & 0.87 & 0.93 & 0.98 & 0.78 & 0.89 & 0.95 & 0.99 |
| | | 100 | 0.77 & 0.87 & 0.93 & 0.98 & 0.78 & 0.89 & 0.95 & 0.99 |
| | | 150 | 0.78 & 0.88 & 0.93 & 0.98 & 0.79 & 0.89 & 0.95 & 0.99 |
| | | 200 | 0.79 & 0.89 & 0.94 & 0.98 & 0.79 & 0.90 & 0.95 & 0.99 |
| 2 | 0.4297 | 25 | 0.76 & 0.85 & 0.91 & 0.96 & 0.79 & 0.90 & 0.95 & 0.99 |
| | | 50 | 0.78 & 0.88 & 0.93 & 0.98 & 0.79 & 0.90 & 0.95 & 0.99 |
| | | 75 | 0.78 & 0.88 & 0.93 & 0.98 & 0.79 & 0.90 & 0.95 & 0.99 |
| | | 100 | 0.78 & 0.88 & 0.94 & 0.98 & 0.79 & 0.90 & 0.95 & 0.99 |
| | | 150 | 0.79 & 0.89 & 0.94 & 0.98 & 0.80 & 0.90 & 0.95 & 0.99 |
| | | 200 | 0.80 & 0.89 & 0.94 & 0.99 & 0.80 & 0.90 & 0.95 & 0.99 |
| 3 | 0.4498 | 25 | 0.77 & 0.86 & 0.91 & 0.97 & 0.80 & 0.90 & 0.95 & 0.99 |
| | | 50 | 0.78 & 0.88 & 0.93 & 0.98 & 0.80 & 0.90 & 0.95 & 0.99 |
| | | 75 | 0.78 & 0.88 & 0.94 & 0.98 & 0.79 & 0.90 & 0.95 & 0.99 |
| | | 100 | 0.79 & 0.88 & 0.94 & 0.98 & 0.79 & 0.90 & 0.95 & 0.99 |
| | | 150 | 0.79 & 0.89 & 0.94 & 0.98 & 0.80 & 0.90 & 0.95 & 0.99 |
| | | 200 | 0.80 & 0.90 & 0.94 & 0.99 & 0.80 & 0.90 & 0.95 & 0.99 |
| 5 | 0.4747 | 25 | 0.78 & 0.87 & 0.92 & 0.97 & 0.80 & 0.90 & 0.95 & 0.99 |
| | | 50 | 0.79 & 0.89 & 0.93 & 0.98 & 0.80 & 0.90 & 0.95 & 0.99 |
| | | 75 | 0.78 & 0.89 & 0.94 & 0.98 & 0.79 & 0.90 & 0.95 & 0.99 |
| | | 100 | 0.79 & 0.89 & 0.94 & 0.98 & 0.79 & 0.90 & 0.95 & 0.99 |
| | | 150 | 0.80 & 0.89 & 0.94 & 0.98 & 0.80 & 0.90 & 0.95 & 0.99 |
| | | 200 | 0.80 & 0.90 & 0.95 & 0.99 & 0.80 & 0.90 & 0.95 & 0.99 |
| 7 | 0.4900 | 25 | 0.78 & 0.87 & 0.92 & 0.97 & 0.80 & 0.90 & 0.95 & 0.99 |
| | | 50 | 0.79 & 0.89 & 0.94 & 0.98 & 0.80 & 0.90 & 0.95 & 0.99 |
| | | 75 | 0.79 & 0.89 & 0.94 & 0.98 & 0.79 & 0.90 & 0.95 & 0.99 |
| | | 100 | 0.79 & 0.89 & 0.94 & 0.98 & 0.79 & 0.90 & 0.95 & 0.99 |
| | | 150 | 0.80 & 0.89 & 0.94 & 0.98 & 0.80 & 0.90 & 0.95 & 0.99 |
| | | 200 | 0.80 & 0.90 & 0.95 & 0.99 & 0.80 & 0.90 & 0.95 & 0.99 |
| 10 | 0.5046 | 25 | 0.78 & 0.87 & 0.93 & 0.97 & 0.80 & 0.90 & 0.95 & 0.99 |
| | | 50 | 0.79 & 0.89 & 0.94 & 0.98 & 0.80 & 0.91 & 0.95 & 0.99 |
| | | 75 | 0.79 & 0.89 & 0.94 & 0.98 & 0.79 & 0.90 & 0.95 & 0.99 |
| | | 100 | 0.79 & 0.89 & 0.94 & 0.98 & 0.79 & 0.90 & 0.95 & 0.99 |
| | | 150 | 0.80 & 0.89 & 0.94 & 0.98 & 0.80 & 0.90 & 0.95 & 0.99 |
| | | 200 | 0.80 & 0.90 & 0.95 & 0.99 & 0.80 & 0.90 & 0.95 & 0.99 |

Table 6: Monte Carlo confidence levels for the E-Gini index.