Gini's Nuclear Family

Rolf Aaberge

Research Department

Statistics Norway

Abstract

The purpose of this paper is to justify the use of the Gini coefficient and two close relatives for summarizing the basic information of inequality in distributions of income. To this end we employ a specific transformation of the Lorenz curve, the scaled conditional mean curve, rather than the Lorenz curve as the basic formal representation of inequality in distributions of income. The scaled conditional mean curve is shown to possess several attractive properties as an alternative interpretation of the information content of the Lorenz curve and furthermore proves to yield essential information of polarization in the population.

Keywords: The scaled conditional mean curve, measures of inequality, the Gini coefficient, the Bonferroni coefficient.

JEL classification: D3, D63.

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Address: Research Department, Statistics Norway, P.O.B. 8131 Dep., N-0033 Oslo, Norway

E-mail: rolf.aaberge@ssb.no

1. Introduction

Empirical analyses of inequality in income distributions are conventionally based on the Lorenz curve. To summarize the information content of the Lorenz curve and to achieve rankings of intersecting Lorenz curves the standard approach is to employ the Gini coefficient in combination with one or two inequality measures from the Atkinson family or the Theil family. However, since the Gini coefficient and Atkinson's and Theil's measures of inequality have distinct theoretical foundations it is difficult to evaluate their capacity as complementary measures of inequality.

By exploiting the fact that the Lorenz curve can be considered analogous to a cumulative distribution function, Aaberge (2000) draw on standard statistical practice to justify the use of the first few moments of the Lorenz curve (LC-moments) as basis for summarizing the information content of the Lorenz curve. However, considered as a group these measures suffer from a drawback since none of them in general turn their attention to the lower part of the income distribution. The reason why the moments of the Lorenz curve in most cases are more sensitive to changes that take place in the central and upper part rather than in the lower part of the income distribution is simply due to the fact that the Lorenz curve has a convex functional form which makes its derivative skew to the left. Thus, even though the first three LC-moments in many cases jointly provide a good description of the inequality in an income distribution it would for informational reasons as well as for the sake of interpretation be preferable to employ a few measures of inequality that also prove to supplement each other with regard to focus on transfers at the lower, the central and the upper part of the income distribution. To this end Section 2 provides arguments for using a specific transformation of the Lorenz curve, the scaled conditional mean curve, rather than the Lorenz curve as basis for introducing and justifying application of a few measures for summarizing inequality in income distributions. The scaled conditional mean curve turns out to possess several useful properties which will be discussed below. Section 3 demonstrates that the moments of the scaled conditional mean curve define a convenient family of inequality measures where the first three moments prove to supplement each other with regard to focus on the lower, the central and the upper part of the income distribution. Section 4 summarizes the paper.

2. The scaled conditional mean curve

Let X be an income variable with cumulative distribution function $F(\cdot)$, density $f(\cdot)$ and mean μ. Let $[0,\infty>$ be the domain of F where $F^{-1}(0)\equiv 0$. The Lorenz curve $L(\cdot)$ for F is defined by

(1) $$L(u)=\frac{1}{\mu}\int_0^u F^{-1}(t)\,dt, \quad 0\le u\le 1,$$

where F^{-1} is the left inverse of F.

The Lorenz curve is concerned with shares of income rather than relative levels of income and differs in that respect from the decile-specific presentation of income inequality which displays decile-specific mean incomes as fractions of the overall mean income. This method of presentation is frequently used by national bureaus of statistics and by researchers dealing with analyzing distributions of income. By introducing a simple transformation of the Lorenz curve we obtain an alternative interpretation of the information content of the Lorenz curve which proves to be closely related to the conventional decile-specific approach mentioned above. To this end we use the scaled conditional mean curve M(·) introduced by Aaberge (1982) and defined by[1]

$$M(u) \equiv \frac{E\left[X \mid X \le F^{-1}(u)\right]}{\mu} = \begin{cases} \dfrac{1}{u\mu}\displaystyle\int_0^u F^{-1}(t)\,dt, & 0 < u \le 1 \\ 0\;, & u = 0. \end{cases} \tag{2}$$

When inserting for (1) in (2) the following simple relationship between the scaled conditional mean curve and the Lorenz curve emerges,

$$M(u) = \begin{cases} \dfrac{L(u)}{u}, & 0 < u \le 1 \\ 0\;, & u = 0, \end{cases} \tag{3}$$

where $M(1) = 1$ and $\lim_{u \to 0}\left(L(u)/u\right) = M(0)$. Thus, formally the scaled conditional mean curve is a representation of inequality that is equivalent to the Lorenz curve.

The scaled conditional mean curve possesses several attractive properties. First, it provides a convenient alternative interpretation of the information content of the Lorenz curve. For a fixed u, M(u) is the ratio between the mean income of the poorest 100u per cent of the population and the overall mean. Thus, the scaled conditional mean curve may also yield essential information on poverty provided that we know the poverty rate. Second, the scaled conditional mean curve of a uniform (0,a) distribution proves to be the diagonal line joining the points (0,0) and (1,1) and thus represents a useful reference line, in addition to the two well-known standard reference lines. The egalitarian reference line, coincides with the horizontal line joining the points (0,1) and (1,1). At the other extreme, when one person holds all income, the scaled conditional mean curve coincides with the horizontal axis except for $u = 1$. Third, the family of scaled conditional mean curves is bounded by the unit square. Therefore visually, there is a sharper distinction between two different scaled conditional mean curves than between the two corresponding Lorenz curves. This distinction appears to be

[1] This ratio was also considered by Nygård and Sandström (1981), but they did not explore its properties as a function that is uniquely determined by the Lorenz curve, whilst Atkinson and Bourguignon (1989) used the numerator as an alternative interpretation of the information provided by the generalized Lorenz curve.

particular visible at the lower parts of the income distributions[2]. As an illustration Figures 1 and 2 give the Lorenz curves and the scaled conditional mean curves of the distributions of average annual earnings in Norway for the periods 1981-1982 and 1986-1987.

[Insert Figures 1 and 2 here]

As can be seen from the scaled conditional mean curves there may be differences in inequality between the lower tails of two distribution functions which may be perceived as negligible when the judgment relies on the plots of the corresponding Lorenz curves. Note, however, that a judgment of the statistical significance of this difference in inequality does not depend on whether we rely on the scaled conditional mean curve or the Lorenz curve. However, the question of whether a difference or change in inequality is large or small is separate from that of statistical significance, and appears to be more easy to deal with when we rely on plots of the scaled conditional mean curve rather than on plots of the Lorenz curve[3].

In contrast to the Lorenz curve, which always is a convex function, the shape of the scaled conditional mean curve proves to be strongly related to the shape of the underlying distribution function. In order to demonstrate this fact observe that the first derivative of M is non-negative and that the second derivative of M is given by

$$M''(u) = -\frac{1}{\mu u^3}\int_0^u \frac{t^2 f'\left(F^{-1}(t)\right)}{f^3\left(F^{-1}(t)\right)}dt, \tag{4}$$

provided that $\left[u^2 / f\left(F^{-1}(u)\right)\right] \to 0$ when $u \to 0+$. The expression (4) for the second derivative of M demonstrates that there is a close relationship between the shape of the distribution function F and the shape of the scaled conditional mean curve. For example, when F is convex, i.e. F is strongly skew to the left, then M is concave. By contrast, when F is concave, i.e. F is strongly skew to the right, then M is convex. Moreover, a symmetric and convex/concave distribution function F implies a concave/convex shape of the corresponding scaled conditional mean curve, whereas a symmetric and concave/convex F implies a convex/concave scaled conditional mean curve. Note that a concave/convex distribution function occurs when there is a tendency of stratification/polarization in

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[2] Atkinson and Bourguignon (1989) brought forward this property to justify the use of the "incomplete mean curve" (the numerator of M) rather than the generalized Lorenz curve.

[3] The estimates of Figures 1 and 2 are based on data of 621 804 persons available from Statistics Norway’s Tax Assessment Files. Thus, sampling errors are of minor importance in this case.

the population[4]. At the extreme the concave/convex (and symmetric) F becomes a two-point distribution function which displays complete polarization.

3. Gini's nuclear family of inequality measures

By observing that the Lorenz curve can be considered analogous to a cumulative distribution function Aaberge (2000) demonstrated that the moments of the Lorenz curve generate the following family of inequality measures

(5) $$D_k(F) = \frac{1}{k}\left(\left(k+1\right)\int_0^1 u^k dL(u) - 1\right), \quad k = 1, 2, \ldots,$$

called the Lorenz family of inequality measures[5], and moreover proved that it is strongly related to a subfamily of the extended Gini Family discussed by Kakwani (1980), Donaldson and Weymark (1980, 1983) and Yitzaki (1983). Alternatively, the members of the Lorenz family may be expressed in terms of the distribution function F in the following way,

(6) $$D_k(F) = \frac{1}{k\mu}\int F(x)\left(1 - F^k(x)\right)dx, \quad k = 1, 2, \ldots$$

Since the Lorenz curve is uniquely determined by its moments we can, without loss of information, restrict the examination of inequality in an income distribution F to the Lorenz family of inequality measures. However, even though we have obtained to reduce the size of the family of inequality measures from the standard infinite non-countable set to a countable set it still contains infinite members. For practical reasons it would be preferable to rely on a few measures of inequality in empirical applications. By drawing on standard statistical practice Aaberge (2000) proposed to use the first few moments of the Lorenz curve as primary quantities for measuring inequality, i.e. D_1, D_2 and D_3 where D_1 is the Gini coefficient. These three measures may jointly give a good summarization of the information provided by the Lorenz curve but suffer from the inconvenience of generally turning their attention to the central and/or the upper part of the income distribution. However, a measure of inequality that primarily focuses on the lower tail can be obtained by introducing an appropriate linear combination of D_1, D_2 and D_3. As will be demonstrated below an alternative and more attractive strategy may be to use the first three moments of the scaled conditional mean curve as primary quantities for measuring inequality in income distributions. The k^{th} order moment of the scaled conditional mean curve for income distribution F, $C_k(F)$, is defined by

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[4] For recent discussions on polarization we refer to Esteban and Ray (1994) and Wolfson (1994).

[5] Note that this is a subfamily of a family of inequality measures that was introduced by Mehran (1976).

(7) $$C_k(F) = \int_0^1 u^k \, d\, M(u).$$

By recalling the properties of M we immediately realize from (7) that the moments of the scaled conditional mean curve $\{C_k : k = 1,2,\ldots\}$ constitute a family of inequality measures with range [0,1]. Thus, without loss of generalization we can restrict the examination of the inequality in F to the moments of the scaled conditional mean curve. The following alternative expression of C_k,

(8) $$C_k(F) = k\int_0^1 u^{k-1}\left(1 - M(u)\right)du, \quad k = 1,2,\ldots$$

demonstrates that C_k for k>1 is adding up weighted differences between the scaled conditional mean curve and its egalitarian line. The mean of M (C_1) is equal to the area between the scaled conditional mean curve and its egalitarian line[6], the horizontal line joining the points (0,1) and (1,1) of Figure 2. The inequality measure C_1 appears to be identical to a measure of inequality that was introduced by Bonferroni (1930) as an alternative to the Gini coefficient, but since then it has for some reason been paid little attention in the economic literature[7]. By inserting for (2) in (8) when k=1 we obtain the following alternative expression for C_1,

(9) $$C_1(F) = -\frac{1}{\mu}\int F(x)\log F(x)\,dx.$$

Now, inserting (2) into (8) when $k = 2$ we find that the second order moment of the scaled conditional mean curve is equal to the Gini coefficient (C_2), whilst an alternative expression of the third order moment of the scaled conditional mean curve is given by (6) for k=2.

Note that $C_{k+1} = D_k$ for $k = 1, 2, \ldots,$ which means that the family $\{C_k : k = 1, 2, \ldots\}$ simply is the Lorenz family of inequality measures extended with the Bonferroni coefficient C_1. This also means that C_1 is uniquely determined by the Lorenz family measures of inequality. The explicit relationship is found by inserting for (2) in (8) when k=1 and by using Taylor-expansion for the term 1/u. Finally, inserting for (5) in the attained expression yields

(10) $$C_1(F) = \sum_{n=1}^{\infty}\sum_{k=1}^{n}\left(-1\right)^{k-1}\binom{n-1}{k-1}\frac{1}{k+1}D_k(F).$$

[6] Note that Eltetö and Frigyes (1968) proposed $M(F(\mu))$ as a measure of inequality. However, this measure is unaffected by transfers between individuals on the same side of the mean, which means that it does not satisfy the Pigou-Dalton transfer principle.

Since C_1, C_2 and C_3 represent the first, the second and the third order moments of the scaled conditional mean curve, they jointly may make up a fairly good summarization of the scaled conditional mean curve as well as of the Lorenz curve[8]. Moreover, as will be demonstrated below C_1 and C_2 complement the information provided by the Gini coefficient by turning particular attention to changes that take place in the lower and upper part of the income distribution. Due to these features of C_1, C_2 and C_3 we will treat them as a group and call them Gini's nuclear family of inequality measures. Thus, Gini's nuclear family of inequality measures can be considered as an adjustment of the group of measures (D_1, D_2 and D_3) discussed by Aaberge (2000) where $C_2=D_1$, $C_3=D_2$ and D_3 is replaced by the Bonferroni coefficient C_1.

Moreover, Aaberge (2000) demonstrated that the Lorenz family of inequality measures as well as the Bonferroni coefficient can be given explicit expressions in terms of social welfare and are members of the "illfare-ranked single-series Ginis" introduced by Donaldson and Weymark (1980) and discussed by Bossert (1990)[9]. It is easily verified that C_1, C_2 and C_3 preserve first-degree Lorenz dominance and thus satisfy the Pigou-Dalton transfer principle. However, to deal with situations where Lorenz curves intersect a more demanding principle than the Pigou-Dalton transfer principle is required. An obvious idea is to introduce a principle that places more emphasis on a given transfer the lower it occurs in the income distribution. Kolm (1976) and Mehran (1976) proposed two alternative versions of such a principle; the principle of diminishing transfers which requires the income difference between receivers and donors to be fixed and the principle of positional transfer sensitivity which requires a fixed difference in ranks between receivers and donors[10]. By applying Theorem 2 in Aaberge (2000) we find that the Bonferroni coefficient assigns more weight to transfer between persons with a given income difference if these incomes are lower than if they are higher provided that the distribution function are strictly logconcave[11]. This class includes the uniform, the exponential, the Pareto, the Gamma, the Laplace, the Weibull and the Wishart distributions. For logconcave distribution functions there are, as were also noted by Heckman and Honoré (1990) and Caplin and Nalebuff (1991), a rising gap between the income of the richest and the average income of those units with income lower than the richest as we move up the income distribution[12], i.e. $x - E(Y \mid Y \leq x)$ is an increasing function of x. Observe that if X and Y are distributed according to F (with mean μ) we have

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[7] For a few exceptions we refer to D'Addario (1936), Nygård and Sandström (1981) and Aaberge (1982, 2000).

[8] Aaberge (2000) demonstrates that C_2 and C_3 also provide essential information on the shape of the income distribution.

[9] See also Sen (1974), Yitzhaki (1979), Weymark (1981), Hey and Lambert (1980), Donaldson and Weymark (1983), Yaari (1987, 1988), Ben Porath and Gilboa (1994) and Aaberge (2001) who have provided alternative characterizations of the Gini coefficient and generalized Gini measures of inequality.

[10] We refer to Mehran (1976), Kakwani (1980) and Zoli (1999) for a discussion of the principle of positional transfer sensitivity.

[11] For a complete characterization of logconcavity, see An (1998).

[12] Note that the income gap is equal to the average poverty gap when x coincides with the poverty line.

(11) $$C_1(F) = \frac{E\left\{X - E\left(Y \middle| Y \le X\right)\right\}}{\mu}$$

which means that the Bonferroni coefficient is equal to the ratio between the mean of these income gaps and the mean income. Consequently, the Bonferroni coefficient assigns more weight to transfers taking place lower down in the distribution for all distributions which are strongly skew to the right and even for some distributions which are strongly skew to the left. Distributions which are strongly skew to the left exhibit a minority of "poor" individuals/households and a majority of "rich" individuals/households.

When the transfer sensitivity of the Bonferroni coefficient is judged according to the principle of positional transfer sensitivity the results of Aaberge (2000) show that the Bonferroni coefficient (C_1) always treats a given transfer of money from a richer to a poorer person to be more equalizing the lower it occurs in the income distribution, provided that the difference in ranks between receivers and donors is the same.

For a discussion of the transfer sensitivity properties of the Gini coefficient (C_2) and the C_3-coefficient we refer to Aaberge (2000). However, for the sake of completeness we summarize the transfer sensitivity properties of the members of Gini's nuclear family of inequality measures in Proposition 1.

PROPOSITION 1. The inequality measures C_1, C_2 and C_3 have the following transfer sensitivity properties,

(i) The Bonferroni coefficient (C_1) satisfies the principle of diminishing transfers for all strictly log-concave distribution functions and the principle of positional transfer sensitivity for all distribution functions.

(ii) The Gini coefficient (C_2) satisfies the principle of diminishing transfers for all strictly concave distribution functions, but does not satisfy the principle of positional transfer sensitivity. In the case of a fixed difference in ranks the Gini coefficient attaches an equal weight to a given transfer irrespective of whether it takes place in the upper , the middle or the lower part of the income distribution.

(iii) The C_3-coefficient satisfies the principle of diminishing transfers for all distribution functions F for which F^2 is strictly concave, but does not satisfy the principle of positional transfer sensitivity. In the case of a fixed difference in ranks the C_3-coefficient assigns more weight to transfers at the upper than at the central and the lower part of the income distribution.

4. Conclusion

This paper proposes to use a specific transformation of the Lorenz curve, called the scaled conditional mean curve, rather than the Lorenz curve as basis for choosing a few summary measures of inequality for empirical applications. The scaled conditional mean curve turns out to possess several attractive properties as an alternative interpretation of the information content of the Lorenz curve and furthermore proves to provide essential information on polarization in the population. The discussion in Section 3 demonstrates that the inequality measures C_1, C_2 and C_3 define the first three moments of the scaled conditional mean curve. Thus, jointly they may give a good summarization of inequality in the scaled conditional mean curve and consequently act as primary quantities for measuring inequality in distributions of income. Moreover, since C_2 is the Gini coefficient and C_1 and C_3 prove to supplement the Gini coefficient with regard to focus on the lower and the upper part of the income distribution, it should be justified that these three measures of inequality form a group that may be denoted Gini's nuclear family.

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Figure 1. Lorentz curves for distributions of average annual earnings in Norway

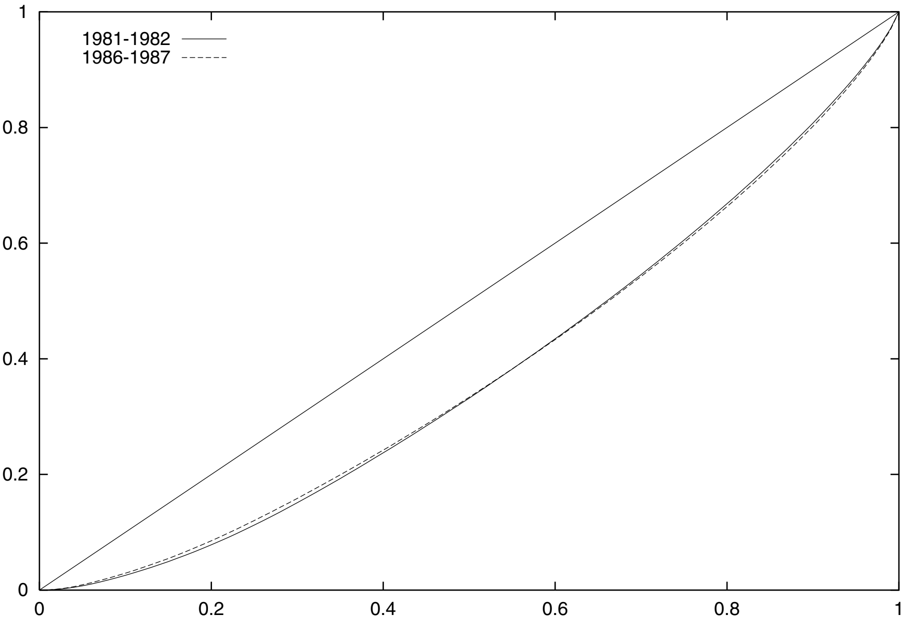

Figure 2. Scaled conditional mean curves for distributions of average annual earnings in Norway